

EIGENVALUES, EIGENVECTORS, AND EIGENSPACES

Defⁿ: Let $L: V \rightarrow V$ be a linear operator on vector space V . A nonzero vector $v \in V$ is an eigenvector with eigenvalue λ when $L(v) = \lambda v$.

Recall that an $n \times n$ matrix determines a linear transformation $L_M: \mathbb{R}^n \rightarrow \mathbb{R}^n$ where $\text{Rep}_{\mathcal{E}_n, \mathcal{E}_n}(L_M) = M$. When we discuss the eigenvalues or eigenvectors of a matrix, we mean the corresponding object for the transformation L_M . Note that the correspondence between $n \times n$ matrices and linear operators on \mathbb{R}^n allows us to work primarily with matrices from now on.

Ex: Let $M = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$. Noting that

$$M \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \text{ we see that}$$

$v = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector of M with eigenvalue $\lambda = 2$. \square

Note that each eigenvalue of M yields a subspace of \mathbb{R}^n .

Prop: Let λ be a scalar and $L: V \rightarrow V$ a linear operator. The set $V_\lambda := \{u \in V : L(u) = \lambda u\}$ is a subspace of V .

pf: We apply the subspace test. In particular, given two elements $u, v \in V_\lambda$ and scalar a , we compute

$$\begin{aligned} L(u + av) &= L(u) + aL(v) && \text{(by linearity of } L) \\ &= \lambda u + a(\lambda v) && \text{(definition of } V_\lambda) \\ &= \lambda u + (a\lambda)v && \text{(vector space axiom)} \\ &= \lambda u + (\lambda a)v && \text{(commute multiplication)} \\ &= \lambda u + \lambda(av) && \text{(vector space axiom)} \\ &= \lambda(u + av) && \text{(scalar distribution)} \end{aligned}$$

Hence $L(u + av) = \lambda(u + av)$ yields $u + av \in V_\lambda$. Note also

$L(0_v) = 0_v = \lambda \cdot 0_v$, so $0_v \in V_\lambda \neq \emptyset$. Hence $V_\lambda \leq V$ as desired. \square

Defⁿ: The spaces $V_\lambda := \{u \in V : L(u) = \lambda u\}$ are eigenspaces.

Observation: If $v \in V_\lambda \cap V_\mu$ and $v \neq 0$, then

$$\lambda v = L(v) = \mu v. \quad \text{Thus } (\lambda - \mu)v = \lambda v - \mu v = 0_v, \quad \text{so}$$

we have $\lambda - \mu = 0$, i.e. $\lambda = \mu$. In particular, eigenspaces of distinct eigenvalues have only the zero vector in common ☺

At this point, we've seen an example and played with some theory. But how do we compute eigenvalues and eigenspaces?

If v is an eigenvector of M with eigenvalue λ , then $Mv = \lambda v$. Subtracting λv we obtain

$$0_v = Mv - \lambda v = Mv - \lambda I v = (M - \lambda I)v.$$

From this we've learned two new facts.

- ① If λ is an eigenvalue of M , then $M - \lambda I$ is singular.
- ② Every eigenvector of M with eigenvalue λ is in $\text{null}(M - \lambda I)$.

For the moment let's focus on condition ①. The matrix $M - \lambda I$ is singular if and only if $\det(M - \lambda I) = 0$.

This simple observation leads us to make a definition.

Defⁿ: The characteristic polynomial of an $n \times n$ matrix M is $P_M(\lambda) := \det(M - \lambda I)$ where λ is a variable.

Now we formalize our observation from above.

Prop: Let M be a matrix. A scalar λ is an eigenvalue of M if and only if λ is a root of P_M .

Point: To compute eigenvalues, we need only compute roots of P_M ☺

Ex: Compute the eigenvalues of $M = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

Sol: First we compute the characteristic polynomial of M .

$$P_M(\lambda) = \det(M - \lambda I) = \det\left(\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$$

$$= \det \begin{bmatrix} 1-\lambda & 1 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & 1-\lambda \end{bmatrix}$$

Cofactor expansion on row 1 \rightarrow

$$= (1-\lambda) \det \begin{bmatrix} -\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix} - \det \begin{bmatrix} 1 & 1 \\ 0 & 1-\lambda \end{bmatrix} + 0$$

2x2 determinant formula \rightarrow

$$= (1-\lambda) (-\lambda(1-\lambda) - 1) - ((1-\lambda) - 0)$$

basic algebra techniques $\left\{ \right.$

$$\begin{aligned} &= -(1-\lambda)(1 + \lambda - \lambda^2) - (1-\lambda) \\ &= -(1-\lambda)(1 + \lambda - \lambda^2 + 1) \\ &= + (1-\lambda)(\lambda^2 - \lambda - 2) \\ &= (1-\lambda)(\lambda-2)(\lambda+1) = -(\lambda+1)(\lambda-1)(\lambda-2) \end{aligned}$$

Hence $P_M(\lambda) = -(\lambda+1)(\lambda-1)(\lambda-2)$ is the characteristic polynomial.

Now we compute the eigenvalues of M by solving $P_M(\lambda) = 0$:

$$P_M(\lambda) = 0 \iff -(\lambda+1)(\lambda-1)(\lambda-2) = 0$$

$$\iff \lambda+1=0 \quad \text{OR} \quad \lambda-1=0 \quad \text{OR} \quad \lambda-2=0$$

$$\iff \lambda = -1 \quad \text{OR} \quad \lambda = 1 \quad \text{OR} \quad \lambda = 2$$

Hence M has eigenvalues $\lambda = -1$, $\lambda = 1$, and $\lambda = 2$. \square

Ex: $A = \begin{bmatrix} 1 & \\ 0 & 1 \end{bmatrix}$ has $P_A(\lambda) = \det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{bmatrix} = (1-\lambda)^2$,

so $\lambda = 1$ is the only eigenvalue of A .

Ex: $B = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ has characteristic polynomial

$$p_B(\lambda) = \det(B - \lambda I) = \det \begin{bmatrix} 1-\lambda & 2 \\ 1 & 1-\lambda \end{bmatrix} = (1-\lambda)^2 - 2.$$

Hence we compute eigenvalues as follows:

$$p_B(\lambda) = 0 \iff (1-\lambda)^2 - 2 = 0$$

$$\iff (1-\lambda)^2 = 2$$

$$\iff 1-\lambda = \pm\sqrt{2}$$

$$\iff -\lambda = -1 \pm \sqrt{2}$$

$$\iff \lambda = 1 \pm \sqrt{2}$$

Thus B has eigenvalues $\lambda = 1 \pm \sqrt{2}$. □

Ex: $C = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}$ has characteristic polynomial

$$p_C(\lambda) = \det(C - \lambda I)$$

$$= \det \begin{bmatrix} 1-\lambda & 3 \\ -1 & 2-\lambda \end{bmatrix}$$

2x2 determinant
formula

$$= (1-\lambda)(2-\lambda) - (-1)(3)$$

Simplify

$$\begin{aligned} &= 2 - 3\lambda + \lambda^2 + 3 \\ &= \lambda^2 - 3\lambda + 5 \end{aligned}$$

"Complete the square"

$$\begin{aligned} &= \left(\lambda^2 - 2\left(\frac{3}{2}\right)\lambda + \left(\frac{3}{2}\right)^2 \right) + \left(5 - \left(\frac{3}{2}\right)^2 \right) \\ &= \left(\lambda - \frac{3}{2} \right)^2 + \left(5 - \frac{9}{4} \right) \\ &= \left(\lambda - \frac{3}{2} \right)^2 + \frac{11}{4} \end{aligned}$$

Hence $p_C(\lambda) = \left(\lambda - \frac{3}{2} \right)^2 + \frac{11}{4}$, which has complex roots!

Indeed, the eigenvalues of C are $\lambda = \frac{3}{2} \pm \frac{\sqrt{11}}{2}i$. □

NB: The last example indicates eigenvalues can be complex!

In the background we're actually working with $L_M: \mathbb{C}^2 \rightarrow \mathbb{C}^2$,
and V_λ is a complex vector space now 😊

At this point we know how to compute eigenvalues via the characteristic polynomial. But what about eigenvectors and eigenspaces? For that we formalize observation ② from earlier.

Prop: Let M be an $n \times n$ matrix with eigenvalue λ . The eigenspace of M associated to λ is $V_\lambda = \text{null}(M - \lambda I)$.

Point: To calculate the eigenspaces of M we must

① Compute $p_M(\lambda)$.

② solve $p_M(\lambda) = 0$ for eigenvalues.

③ For each eigenvalue λ compute $\text{null}(M - \lambda I)$.

Ex: Let $M = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Then the characteristic polynomial

$$p_M(\lambda) = \det \begin{bmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix} = (1-\lambda)^2 - 1 = \lambda(\lambda-2). \text{ Thus}$$

M has eigenvalues $\lambda = 0$ and $\lambda = 2$. We must now compute eigenspaces separately via $V_\lambda = \text{null}(M - \lambda I)$.

$$\underline{\lambda = 0}: M - 0I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \text{ has } \text{RREF}(M - 0I) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix},$$

$$\text{so } \begin{bmatrix} x \\ y \end{bmatrix} \in \text{null}(M - 0I) \iff x + y = 0 \iff x = -y.$$

Hence $\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ is a basis for $V_0 = \text{null}(M - 0I)$.

$$\underline{\lambda = 2}: M - 2I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \text{ has } \text{RREF}(M - 2I) = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix},$$

$$\text{so } \begin{bmatrix} x \\ y \end{bmatrix} \in \text{null}(M - 2I) \iff x - y = 0 \iff x = y.$$

Hence $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ is a basis for $V_2 = \text{null}(M - 2I)$.

Thus $V_0 = \text{span} \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ and $V_2 = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

Ex: Compute the eigenspaces of $M = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

Sol: Earlier we computed eigenvalues $\lambda = -1, 1, 2$.

$\lambda = -1$: $\text{RREF}(M + I) = \text{RREF} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Hence $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \text{null}(M + I) \Leftrightarrow \begin{cases} x = 0 \\ y + z = 0 \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ y = -t \\ z = t \end{cases}$ yields $V_{-1} = \text{span} \left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$.

$\lambda = 1$: $\text{RREF}(M - I) = \text{RREF} \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Hence $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \text{null}(M - I) \Leftrightarrow \begin{cases} x + z = 0 \\ y = 0 \end{cases} \Leftrightarrow \begin{cases} x = -t \\ y = 0 \\ z = t \end{cases}$ yields $V_1 = \text{span} \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

$\lambda = 2$: $\text{RREF}(M - 2I) = \text{RREF} \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$.

Hence $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \text{null}(M - 2I) \Leftrightarrow \begin{cases} x - z = 0 \\ y - z = 0 \end{cases} \Leftrightarrow \begin{cases} x = t \\ y = t \\ z = t \end{cases}$ yields $V_2 = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

This finishes the computation of eigenspaces of M . □

Ex: Compute the eigenspaces of $F = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$.

Sol: Characteristic polynomial $p_F(\lambda) = \det \begin{bmatrix} -\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix} = -\lambda(1-\lambda) - 1 \cdot 1 = \lambda^2 - \lambda - 1$

has roots $\lambda = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)} = \frac{1 \pm \sqrt{5}}{2}$ by the quadratic formula.

Hence we compute the eigenspaces for these eigenvalues below.

$\lambda = \frac{1+\sqrt{5}}{2}$: We compute an echelon form of $F - \lambda I$:

$$\begin{bmatrix} -\frac{1+\sqrt{5}}{2} & 1 \\ 1 & 1 - \frac{1+\sqrt{5}}{2} \end{bmatrix} \rightsquigarrow \begin{bmatrix} -1 - \sqrt{5} & 2 \\ 2 & 1 - \sqrt{5} \end{bmatrix} \rightsquigarrow \begin{bmatrix} 2 & 1 - \sqrt{5} \\ 0 & 0 \end{bmatrix}$$

Hence $\begin{bmatrix} x \\ y \end{bmatrix} \in \text{null}(F - \lambda I) \Leftrightarrow 2x + (1 - \sqrt{5})y = 0$

$$\Leftrightarrow \begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} 1 - \sqrt{5} \\ -2 \end{bmatrix} \quad \text{Some } t$$

We thus have $V_{\frac{1+\sqrt{5}}{2}} = \text{span} \left\{ \begin{bmatrix} 1 - \sqrt{5} \\ -2 \end{bmatrix} \right\}$.

$\lambda = \frac{1-\sqrt{5}}{2}$: We compute an echelon form for $F - \lambda I$:

$$\begin{bmatrix} -\frac{1-\sqrt{5}}{2} & 1 \\ 1 & 1-\frac{1-\sqrt{5}}{2} \end{bmatrix} \rightsquigarrow \begin{bmatrix} -1+\sqrt{5} & 2 \\ 2 & 1+\sqrt{5} \end{bmatrix} \rightsquigarrow \begin{bmatrix} 2 & 1+\sqrt{5} \\ 0 & 0 \end{bmatrix}$$

Hence $\begin{bmatrix} x \\ y \end{bmatrix} \in \text{null}(F - \lambda I) \iff 2x + (1+\sqrt{5})y = 0$

$$\iff \begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} 1+\sqrt{5} \\ -2 \end{bmatrix} \quad \text{some } t$$

This we have $V_{\frac{1-\sqrt{5}}{2}} = \text{Span} \left\{ \begin{bmatrix} 1+\sqrt{5} \\ -2 \end{bmatrix} \right\}$.

□

Ex: Compute the eigenspaces of $M = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$.

Sol: Characteristic polynomial computation yields

$$P_M(\lambda) = \det \begin{bmatrix} 2-\lambda & -1 \\ 1 & 2-\lambda \end{bmatrix} = (2-\lambda)^2 - (-1) = (\lambda-2)^2 + 1$$

which has roots $\lambda = 2 \pm i$, two complex eigenvalues.

$\lambda = 2+i$: $\text{RREF}(M - (2+i)I) = \text{RREF} \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} = \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix}$,

$$\text{so } \begin{bmatrix} x \\ y \end{bmatrix} \in \text{null}(M - \lambda I) \iff x - iy = 0 \iff \begin{cases} x = it \\ y = t \end{cases}$$

and $V_{2+i} = \text{Span} \left\{ \begin{bmatrix} i \\ 1 \end{bmatrix} \right\}$ as a complex vector space.

$\lambda = 2-i$: $\text{RREF}(M - (2-i)I) = \text{RREF} \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} = \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix}$,

$$\text{so } \begin{bmatrix} x \\ y \end{bmatrix} \in \text{null}(M - \lambda I) \iff x + iy = 0 \iff \begin{cases} x = -it \\ y = t \end{cases}$$

and $V_{2-i} = \text{Span} \left\{ \begin{bmatrix} -i \\ 1 \end{bmatrix} \right\}$ as a complex vector space.

NB: The previous examples had all eigenvalues distinct, so this was somewhat special. Indeed, the next few examples are more generic...

Ex: Compute the eigenspaces of $M = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & 1 \end{bmatrix}$.

Sol: $p_M(\lambda) = \det(M - \lambda I)$

$$= \det \begin{bmatrix} 1-\lambda & 0 & 2 \\ 0 & 3-\lambda & 0 \\ 2 & 0 & 1-\lambda \end{bmatrix}$$

$$= (1-\lambda) \det \begin{bmatrix} 3-\lambda & 0 \\ 0 & 1-\lambda \end{bmatrix} - 0 + 2 \det \begin{bmatrix} 0 & 3-\lambda \\ 2 & 0 \end{bmatrix}$$

$$= (1-\lambda)((3-\lambda)(1-\lambda) - 0) + 2(0 - 2(3-\lambda))$$

$$= (3-\lambda)((1-\lambda)^2 - 4)$$

$$= -(\lambda-3)((\lambda-1)^2 - 2^2)$$

$$= -(\lambda-3)(\lambda-3)(\lambda+1)$$

$$= -(\lambda+1)(\lambda-3)^2$$

\therefore have eigenvalues $\lambda = -1$, $\lambda = 3$.

$\lambda = -1$: $\text{RREF}(M + I) = \text{RREF} \begin{bmatrix} 2 & 0 & 2 \\ 0 & 4 & 0 \\ 2 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$

Hence $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \text{null}(M + I) \Leftrightarrow \begin{cases} x + z = 0 \\ y = 0 \end{cases} \Leftrightarrow \begin{cases} x = -z \\ y = 0 \\ z = t \end{cases}$

yields $V_{-1} = \text{null}(M + I) = \text{span} \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$

$\lambda = 3$: $\text{RREF}(M - 3I) = \text{RREF} \begin{bmatrix} -2 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$

So $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \text{null}(M - 3I) \Leftrightarrow x - z = 0 \Leftrightarrow \begin{cases} x = z \\ y = s \\ z = t \end{cases} \Leftrightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$
Some s, t

Hence $V_3 = \text{null}(M - 3I) = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$

In closing note $\dim(V_{-1}) = 1$ while $\dim(V_3) = 2$.



Ex: Compute eigenspaces of $M = \begin{bmatrix} \pi & 1 & 0 \\ 0 & \pi & 0 \\ 0 & 0 & \pi \end{bmatrix}$.

Sol: $p_M(\lambda) = \det(M - \lambda I) = \det \begin{bmatrix} \pi - \lambda & 1 & 0 \\ 0 & \pi - \lambda & 0 \\ 0 & 0 & \pi - \lambda \end{bmatrix} = (\pi - \lambda)^3$.

Hence we have one eigenspace, for eigenvalue $\lambda = \pi$.

$\lambda = \pi$: $\text{RREF}(M - \pi I) = \text{RREF} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \text{null}(M - \pi I) \Leftrightarrow y = 0 \Leftrightarrow \begin{cases} x = s \\ y = 0 \\ z = t \end{cases} \Leftrightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = s e_1 + t e_3$$

Hence $V_\pi = \text{span}\{e_1, e_3\}$. □

Note that the dimensions of the eigenspaces were somewhat off-the-walls in the previous few examples. Indeed, we will want to study this somewhat closely for what is to come.

To begin, let's have a definition.

Defⁿ: Let α be an eigenvalue of M .

- ① The algebraic multiplicity of α is the power of $(\lambda - \alpha)$ present in the factorization of $p_M(\lambda)$.
- ② The geometric multiplicity of α is the dimension of V_α .

First we make a simple observation.

Prop: Let α be an eigenvalue of M . The geometric multiplicity of α is at least 1 and at most the algebraic multiplicity of α .

Q: Why care?

A: Before we saw $V_\alpha \cap V_\beta = \{0_v\}$ unless $\alpha = \beta$. This implies that if $B_\alpha \subseteq V_\alpha$ and $B_\beta \subseteq V_\beta$ are bases, then $B_\alpha \cup B_\beta$ is independent in V . As such, geometric multiplicity will tell us if V has a basis of eigenvectors...

Prop: Let M be an $n \times n$ matrix.

- ① the degree of $P_M(\lambda)$ is n .
- ② \mathbb{R}^n has a basis of eigenvectors of M if and only if the geometric multiplicity of every eigenvalue is equal to its algebraic multiplicity.

Recall that matrices A and B are similar when there is an invertible matrix P such that $B = P^{-1}AP$. We say matrix M is diagonalizable when there is a diagonal matrix D which is similar to M .

Prop (Diagonalizability Criterion) Let M be an $n \times n$ matrix.

The following are equivalent.

- ① M is diagonalizable.
- ② Each eigenvalue of M has equal algebraic and geometric multiplicity.
- ③ \mathbb{R}^n has a basis B in which every vector of B is an eigenvector of M .

Construction: to diagonalize M :

- ① Compute $P_M(\lambda)$ and eigenvalues of M .
- ② Compute a basis of the eigenspace of each eigenvalue.
↳ If $\dim(V_\alpha)$ is less than the algebraic multiplicity of α , then STOP (it's not possible).

- ③ Consider $B = B_{\lambda_1} \cup B_{\lambda_2} \cup \dots \cup B_{\lambda_k}$ where $\lambda_1 < \lambda_2 < \dots < \lambda_k$ are the eigenvalues of M and B_{λ_i} is a basis of V_{λ_i} for all i .

- ④ Let $P = \text{Rep}_{B, E_n}(\text{id})$, so $P^{-1} = \text{Rep}_{E_n, B}(\text{id})$.

😊 Then $D = P^{-1}MP$ is diagonal (if step 2 did not fail).